On the equivalence between SUE and fixed-point states of day-to-day assignment processes with serially-correlated route choice

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Abstract

We propose a new theoretical framework of deterministic processes of traffic assignment able to include day-to-day correlation of the random terms. According to the prevailing interpretation of random utility models, random terms are regarded as individual specific. Correlation is justified by persistence of habits and unobservables. The framework includes the deterministic sequence of systematic utilities based on a learning filter, the stochastic process of the random terms based on a stationary autoregressive structure with i.i.d. Gumbel or multivariate normal one-day marginals, and the resulting stochastic process of choice which generally is not Markov. The fixed point states of the choice process are the classical logit and probit Stochastic User Equilibrium (SUE). The linkage between flows at any day and transition flows is made explicit, and, by this, a new perspective on the interpretation of SUE is opened. SUE is the condition where, at macro level, the observed route flows do not change across days, while, at micro level, individuals change route. Only if random terms are unchanged no individual changes route. Transition flows at SUE are symmetric, i.e. the number of shifters from path i to path j equals the number of shifters from path j to path i. The insights are illustrated by numerical examples related to a two-link and a five-link network.

Keywords: traffic assignment; deterministic process; stochastic user equilibrium; extremal process; Gaussian process; logit; probit.

1 Introduction

Day-to-day dynamics, whereby the evolution over days of travel choice and traffic congestion is linked through a learning model based on driver's past experience, has attracted considerable attention in the last decades. Day-to-day dynamics can be formulated within a continuous and a discrete time framework (Cantarella and Watling, 2016a, have recently provided a unifying framework), the latter only is considered in this paper.

Deterministic processes were the first to be tackled. A distinctive feature is that route flows are regarded as deterministic variables. After the initial contribution by Horowitz (1984), research has dealt, in particular, with the relationship with Stochastic User Equilibrium (SUE) and stability analysis (Cantarella, 1993; Watling, 1999; Watling and Hazelton, 2003; Bie and Lo, 2010; Zhao and Orosz, 2014; Xiao and Lo, 2015; Cantarella and Watling, 2016b; Guo and Huang, 2016). The solutions of the fixed point problem termed SUE, including logit and probit SUE, are the fixed point states of the assignment process. A slightly different approach has been developed by Guo et al. (2013) who consider the link flow dynamics in lieu of the route flow dynamics.

Other authors have considered route flows as stochastic variables. This alternative assumption can be justified on different grounds as discussed in Watling (2002a). One of the approaches followed is that of stochastic processes, which considers the day-to-day evolution of the probability distribution of route flows regarded as integer variables (Cascetta, 1989; Davis and Nihan, 1993; Cantarella and Cascetta, 1995; Watling, 1996; Watling and Hazelton, 2003; Hazelton and Watling, 2004; Watling and Cantarella, 2015; Parry et al., 2016). The other approach is the generalisation of SUE proposed by Watling (2002a, 2002b), in which stochasticity in both flows and costs is endogenous to the equilibration process.

The present paper deals with deterministic processes. The paper aims to formulate a theory of deterministic assignment processes able to take explicitly into account the day-to-day correlation of the random terms of the route choice model, and to derive a characterisation of SUE within this theory.

Since Daganzo and Sheffi (1977), SUE has been used for its potential to overcome the perfect knowledge assumption of classical Deterministic User Equilibrium (DUE) derived from Wardrop's first principle (Wardrop, 1952). Imperfect knowledge of the network, as it occurs in the absence of Advanced Traveller Information Systems (ATIS), and the associated heterogeneity in the perception of route travel times justify SUE providing strictly positive flows for all routes. A number of contributions have tackled formulation and algorithmic aspects for SUE, many others have used SUE for applications without and with the presence of ATIS (having assumed the perception variance to be related to the information quality), or proposed extensions to SUE. SUE is directly formulated as fixed-point problem, equivalence has been proved with minimisation and variational-inequality problems (see, respectively, Fisk, 1980, and Lo and Szeto, 2001). It is beyond the scope of the paper to provide a full account of literature related to algorithmic aspects, applications and extensions of SUE. To name a few, Powell and Sheffi (1982) and Liu et al. (2009) on the Method of Successive Averages (MSA). Nielsen et al. (2002) for a wide-scale application, Huang and Li (2006) and Huang et al. (2011) for applications in the presence of ATIS, Fan and Liu (2010) on network protection. Uchida et al. (2007) on extension to multimodal route choice, Unnikrishnan and Waller (2009) on extension to en-route choices, Karoonsoontawong and Lin (2015) on combined destination and route choices.

The research here is motivated by the desire to address a few limitations of the classical framework of assignment processes (a comprehensive presentation is in Cantarella and Cascetta, 1995; see Appendix 1 of the present paper for a formal review of the relevant assumptions). This framework builds on the formulation of a choice updating process with an implicit Markovian assumption. Probabilities of choosing a path at a given day conditional on the choice of the path in the previous day are at the cornerstone of the framework (these probabilities are called transition probabilities and the associated matrix transition matrix). The Markov assumption makes it possible to derive the probability of the sequence of choices. Based on Markov chain theory (Norris, 1997), sequence probabilities are given by a product of conditional probabilities (factoring property). In addition, in the classical framework of day-to-day dynamics, conditional probabilities are usually assumed to be independent of the path chosen the previous day. This eventually implies day-to-day independence of the random terms and no state dependence of the systematic utilities.

The random term independence assumption is discussed in Watling and Hazelton (2003) and Watling and Cantarella (2013). They observe that the assumption is somewhat unrealistic as one may expect that traveller's personal preferences for a particular route would persist from day to day. They remark that different approaches have been proposed to deal with this problem. Watling and Cantarella (2013) discuss, in particular, proportional re-routing, whereby, based on an exponential filter, only a fraction of travellers re-route each day according to the choice model. This approach is found in several papers (Cantarella, 1993; Cantarella and Cascetta, 1995; Hazelton and Polak, 1997; Polak and Hazelton, 1998; Cantarella and Watling, 2016b). Another approach, suggested by Polak and Hazelton (1998), considers that re-routing occurs only if the traveller perceives an alternative that has utility at least s units higher than her present route. This approach has similarities with state-dependence, whereby the systematic utility of the present route is assigned an extra-utility to represent the additional psychological cost of switching, as in the model by Cascetta and Cantarella (1991).

The framework that is proposed in the present paper moves a step backwards: instead of starting from a choice updating model with an implicit Markovian assumption, it models the random term day-to-day updating process that occurs together with the adjustment of systematic utilities. It then derives the consequent choice process which, in general, is not Markov[1].

[1]Numerical tests by the author have confirmed that, for general correlation patterns of the random terms, choice sequence probabilities do not satisfy the factoring property, which is a necessary and sufficient condition of Markov chains. Dagsvik (1983, 1988) appears to be the only who has investigated discrete choice processes. He finds conditions on the utility process that guarantee that the attendant choice process

The random term updating process is able to deal with habit persistence, because it assumes day-to-day statistical dependence. The prevailing interpretation of random utility models regards the random terms as individual specific, i.e. random terms account for inter-personal heterogeneity (since McFadden, 1981). Each individual, when faced with a sequence of choices, may change her random term vector. Independence across choices is an un-necessarily restrictive assumption, since some degree of serial correlation is likely to exist because of temporally persistent unobservables and tastes. At the other extreme, we have the perfect correlation assumption, whereby random terms are unchanged across choices. Realistically, day-to-day correlation will lie somewhere between independence and perfect correlation.

Day-to-day dependence of the random terms has impacts on the joint probability, i.e. the probability of choosing alternative i the day before and alternative j the day after. We will refer to this probability as transition, or switching, probability, consistently with the convention adopted in discrete choice literature, in particular by De Palma and Kilani (2005, 2013).

Transition probabilities in discrete choice models are investigated in a stream of research that is ultimately concerned with the implications on welfare measures. De Palma and Kilani (2005 and 2013) provide analytic functional forms of transition probabilities, for logit and general random utility models, under the assumption that random terms remain unchanged. Delle Site and Salucci (2013) extend the framework provided in De Palma and Kilani (2013) to imperfect before-after correlation and review numerical methods for computation. In a subsequent paper, Delle Site and Salucci (2015) provide analytic functional forms of transition probabilities for general day-to-day correlation patterns in the case of logit.

In the conventional day-to-day dynamics framework, when conditional probabilities are assumed independent of the choice made in the previous day, re-routing is quantified by transition probabilities given, in the light of the independence assumption, simply by the product of the probability of choosing alternative i the day before times the probability of choosing alternative j the day after. Transition probabilities have a different value if the independence-across-days assumption is removed.

The paper recasts the theoretical framework of deterministic processes. At the lower level, we have the sequence of systematic utilities, based on the learning filter, and the stationary stochastic process of the random terms. The paper will show different options that can be used to define this stochastic process. These will allow to derive logit and probit SUE from the new framework. At the upper level, as a consequence of the lower levels, we have the stochastic process of route choice which gives rise to route flows.

The paper presents this new framework and shows the noteworthy implications in terms of interpretation of classical logit and probit SUE. Transition probabilities play a key role in this respect. The organisation is as follows. Section 2 presents notation and assumptions.

is Markov.

Section 3 presents the consequent properties of assignment processes and the derivation of SUE. Section 4 provides two illustrative numerical examples.

2 Notation and assumptions

2.1 Network assumptions

Let $G = (N, A)$ be a strongly connected transportation network, with N and A being, respectively, the set of nodes and directed links. Let a be the link index. Origins (O) and destinations (D) constitute a subset of N. Let r=1,...,R be the OD pair index. Let $i = 1, ..., J^r$ be the index of simple paths (routes) of OD pair r.

For each path $i = 1, ..., J^r$, r=1,...,R, F_i^r denotes the corresponding path flow. We denote by z_a the flow on link $a \in A$. The link flows are obtained from the path flows by:

$$z_a = \sum_{r=1}^{R} \sum_{i=1}^{J^r} \delta_a^{i,r} \cdot F_i^r \quad a \in A \tag{1}$$

where $\delta_a^{i,r}$ is the element of the link-path incidence matrix whose value is 1 if path i of OD pair r includes link a, is 0 otherwise.

The demand flow of OD pair r is denoted by q^r.We have the demand constraints:

$$q^r = \sum_{i=1}^{J^r} F_i^r \quad r = 1, ..., R \tag{2}$$

The feasible path flows are all the non-negative F_i^r satisfying the demand constraints.

Let T_i^r denote the travel time on path i of OD pair r. Let T_a denote the travel time on link a. The link travel times are continuous functions of the link flows: $T_a = T_a (z_a,\ a \in A)$. The path travel times are obtained from the link travel times by the standard link-additive model:

$$T_i^r = \sum_{a \in A} \delta_a^{i,r} \cdot T_a (z_a,\ a \in A) \quad i = 1, ..., J^r, \quad r = 1, ..., R \tag{3}$$

2.2 Behavioural assumptions

At day t_n, the individuals of each OD pair perceive a utility $u_i^r(t_n)$ on each path. This path utility is a random variable given by the sum of a systematic, i.e. deterministic, component $v_i^r(t_n)$ and a random term $\varepsilon_i^r(t_n)$:

$$u_i^r(t_n) = v_i^r(t_n) + \varepsilon_i^r(t_n) \quad i = 1, ..., J^r \quad r = 1, ..., R \tag{4}$$

The random terms summarise factors that are unobserved by the modeller. The random terms are interpreted as individual specific, thus accounting for inter-individual variability.

For ease of notation we omit the individual index. They are subject to day-to-day variability according to the continuous state space vector stochastic process:

$$\varepsilon_i^r(t_1), ..., \varepsilon_i^r(t_n) \quad i = 1, ..., J^r \quad r = 1, ..., R \tag{5}$$

The choice is made on the basis of the utilities. Therefore, the path alternative $\Im^r(t_n)$ that is selected at day t_n is a random variable, because the utilities are random. The selected alternative is subject to day-to-day variability according to the discrete state space stochastic process:

$$\Im^r(t_1), ..., \Im^r(t_n) \quad r = 1, ..., R \tag{6}$$

We have three behavioural assumptions: the first relates to the systematic utility updating model, the second to the random term updating model, the third to the choice model. This is illustrated diagrammatically in Figure 1.

2.2.1 Systematic utilities

Assumption A1 - Systematic utility updating model. *The path systematic utility depends on two attributes: a component related to the travel times experienced in the previous day, and a component related to the monetary cost paid to use the path:*

$$v_i^r(t_n) = \beta_T \cdot T_i^r(t_{n-1}) + \beta_C \cdot C_i^r \tag{7}$$

where β_T, β_C are estimation coefficients. The experienced travel time is deterministic and dependent on link flows obtained from path flows:

$$T_i^r(t_{n-1}) = \omega_i^r(F_i^r(t_{n-1}),\ i = 1, ..J,\ r = 1, ..R) \quad i = 1, ..., J;\ r = 1, ..., R \tag{8}$$

The assumption assumes that users are not provided with real-time information.

Based on assumption A1, systematic utilities are characterised by a deterministic sequence, because flows $F_i^r(t_n)$ are regarded as deterministic quantities.

2.2.2 Random terms

We assume for the random terms a strictly stationary[2] autoregressive structure whereby, consistently with the systematic utility updating model, the random term at day t_n is obtained from the combination of the random term at day t_{n-1} and of an independent random term.

Stationarity is a reasonable approximation of the real behaviour when the phenomenon evolves smoothly rather than by abrupt changes. The assumption is also motivated by

[2] In words, strict stationarity refers to the condition where the distribution across one or more consecutive days is invariant with respect to time axis translations.

the need for maintaining consistency with the classical framework of assignment processes which postulates a time-homogeneous one-day marginal distribution.

The mathematical operators involved in the combination depend on the one-day marginal distribution that is postulated. We show hereafter that the operator is the maximum function for a i.i.d. Gumbel marginal (the one of the logit model), and it is the sum for a multivariate normal marginal with general variance-covariance matrix (the one of the probit model).

The distribution of the random terms over a single day are of interest for the computation of path choice probabilities. The logit model is of interest mainly because of the closed-form probabilities. The independence across alternatives assumption is a limitation in the light of overlapping routes. However, it is possible to deal with this problem by appropriately modifying the systematic part of the utility, as in the c-logit model proposed by Cascetta et al. (1996), and the path-size logit proposed by Ben-Akiva and Bierlaire (1999). The probit model requires simulation to compute choice probabilities, but it is able to represent the similarity occurring in overlapping routes at the level of the random terms by appropriately structuring the variance-covariance matrix.

In the following parts of this section we omit for ease of notation the OD pair index.

Assumption A2a - Random term updating model, extremal vector process. *The vector stochastic process of the random terms over consecutive days is strictly stationary with marginal distribution at any given day i.i.d. Gumbel; for each alternative independently, the error term $\varepsilon_i(t_n)$ is generated according to the autoregressive structure:*

$$\varepsilon_i(t_n) = \max\left\{\varepsilon_i(t_{n-1}) + \ln\phi; \tilde{\varepsilon}_i\left(t_n\right) + \ln\left(1-\phi\right)\right\} \quad i = 1,..J \tag{9}$$

where $\tilde{\varepsilon}_i(t_n)$ is a standard Gumbel distributed variable[3]*, generated independently from $\varepsilon_i(t_{n-1})$, and $0 \leq \phi \leq 1$ is a parameter controlling day-to-day correlation.*

This stationary stochastic process was investigated in a number of papers by Tiago de Oliveira (a review is in Tiago de Oliveira, 1980). We summarise hereafter a few properties whose proofs can be found in the literature.

The process has a standard Gumbel one-day marginal (recall the max-stability of the Gumbel distribution). The associated distribution of the random terms over two consecutive days is the so-called bi-extremal distribution. The bi-extremal distribution is also referred to as bivariate Gumbel Type C (Kotz et al., 2000). The cumulative distribution function (c.d.f.) is:

$$\begin{aligned} H_{\varepsilon_{i_{n-1}}\varepsilon_{i_n}}\left[\varepsilon_i(t_{n-1}), \varepsilon_i(t_n)\right] = \exp(-\max[\exp\left(-\varepsilon_i(t_{n-1})\right) \\ + (1-\phi)\cdot\exp\left(-\varepsilon_i(t_n)\right); \exp\left(-\varepsilon_i(t_n)\right)]) \quad i = 1,..J \end{aligned} \tag{10}$$

[3] A standard Gumbel has mean equal to γ, the Euler constant, and standard deviation equal to $\pi/\sqrt{6}$.

The Pearson's product-moment correlation coefficient ρ is given by:

$$\rho\left(\phi\right)=-\frac{6}{\pi^{2}}\cdot\int_{0}^{\phi}\frac{\ln s}{1-s}ds \tag{11}$$

The values of ϕ for given ρ are obtained from solving eqn (11). In particular, for $\rho = 0$ we have $\phi = 0$, and for $\rho = 1$ we have $\phi = 1$.

From eqn (9), as it is immediately verified, when $\rho = 0$, i.e. zero correlation, we have that $\varepsilon_i(t_{n-1})$ and $\varepsilon_i(t_n)$ are independent. This is because, in eqn (9), as ϕ approaches zero from the right the limit of $ln\phi$ is $-\infty$. When $\rho = 1$, i.e. perfect correlation, we have $\varepsilon_i(t_{n-1}) = \varepsilon_i(t_n)$. This is because, in eqn (9), as ϕ approaches one from the left the limit of $ln(1-\phi)$ is $-\infty$.

The c.d.f. in eqn (10) is non differentiable, therefore to obtain the probability density function (p.d.f.) appropriate partitioning of the space is needed. The joint distribution of the two vectors $\varepsilon_i(t_{n-1})$ and $\varepsilon_i(t_n)$, $i = 1, ..J$, will be obtained on the basis of statistical independence across alternatives, i.e. the c.d.f. and p.d.f. are simply given by the product of the bivariate alternative-specific c.d.f and p.d.f..

The bi-extremal distribution is characterised by a particular interpretation when the random terms are regarded as individual specific. Consider a population of individuals with identical systematic part of the utilities. Based on eqn (9), there is an individual-specific minimum perception error in each choice pair equal to $\tau_i = \varepsilon_i(t_{n-1}) + \ln\phi$. In fact, at day t_n, the perception error is never lower than τ_i. At day t_{n-1}, the perception error is $\varepsilon_i(t_{n-1}) > \tau_i$ because the logarithm of ϕ is negative when $\phi \in [0, 1]$.

Figure 2, 3 and 4 show the day-to-day univariate random process defined by eqns (9) for three values of the parameter ϕ controlling day-to-day correlation ($\phi = 0,\ 0.5,\ 1$).

Assumption A2b - Random term updating model, Gaussian vector process. *The vector stochastic process of the random terms over consecutive days is strictly stationary with marginal (joint) distribution at any given day multivariate normal; for each alternative, the error term $\varepsilon_i(t_n)$ is generated according to the autoregressive structure:*

$$\varepsilon_i\left(t_n\right)=\rho\cdot\varepsilon_i\left(t_{n-1}\right)+\sqrt{1-\rho^{2}}\cdot\widetilde{\varepsilon}_i\left(t_n\right)\qquad i=1,..J \tag{12}$$

where the vector $\varepsilon_i\left(t_{n-1}\right)$, $i = 1, ..J$, is multivariate normal with zero means, unit variances and variance-covariance matrix Ξ, the vector $\varepsilon_i\left(t_{n-1}\right)$, $i = 1, ..J$, and the vector $\widetilde{\epsilon}_i\left(t_n\right)$ $i = 1, ..J$, are independent[4]*, the vector $\widetilde{\epsilon}_i\left(t_n\right)$, $i = 1, ..J$, is multivariate normal with zero means, unit variances and variance-covariance matrix Ξ and is independent across days, ρ is the within-alternative day-to-day correlation coefficient.*

It can be proved that, with this structure, the vector $\varepsilon_i\left(t_n\right)$, $i = 1, ..., J$, of the error terms at day t_n and the vector $\varepsilon_i\left(t_{n-1}\right)$, $i = 1, ..., J$, of the error terms at day t_{n-1} have

[4]We recall that if two vectors of random variables are independent, the respective components are pairwise independent.

an identical marginal (joint) distribution. The proof consists in showing that both the normal random vectors have identical variance-covariance matrix Ξ. The following lemma known from the theory of Gaussian random vectors (Brockwell and Davis, 1991) needs to be applied: let $\mathbf{X}$ be a n-dimensional normal random vector with means μ and variance-covariance matrix Υ; if C is an $m \times n$ matrix, then $C{\cdot}\mathbf{X}$ is normal with means $\mathrm{C}{\cdot}\mu$ and variance-covariance matrix $C \cdot \Upsilon \cdot C'$.

Notice that the stationarity of the vector process defined in eqns (12) does not require independence across path alternatives but holds for any pattern of correlation defined in the variance-covariance matrix Ξ. Notice also that all three vectors $\varepsilon_i(t_n)$, $\varepsilon_i(t_{n-1})$ and $\widetilde{\varepsilon}_i(t_n)$, $i = 1, ..J$, have the same variance-covariance matrix Ξ.

The procedure to obtain draws from a multivariate normal is as follows (Scheuer and Stoller, 1962). Let $\mathbf{X}$ have a multivariate normal distribution with zero means and variance-covariance matrix Υ. Let the lower triangular matrix L be obtained by the Cholesky decomposition $L \cdot L' = \Upsilon$. Then, given the random vector $\mathbf{Z}$ with independent standard normal components, the draws of $\mathbf{X}$ are obtained using the transformation: $\mathbf{X} = L \cdot \mathbf{Z}$.

Figure 5, 6 and 7 show the day-to-day univariate random process defined by eqns (12) for three values of the day-to-day correlation coefficient ρ ($\rho = 0,\ 0.5,\ 1$).

2.2.3 Choice

Assumption A3 - Choice model. *At any given day, each user selects the path $\Im^r(t_n)$ with the highest perceived utility:*

$$\Im^r(t_n):\ \ u^r_{\Im}(t_n) = \max_{i=1,\ldots,J^r} u^r_i(t_n) \qquad r = 1, ..., R \tag{13}$$

The stochastic process at the level of choice is characterised statistically by a probability mass function (p.m.f.) which is consequence of the assumptions A1, A2a or A2b, and A3.

The p.m.f. can be defined with reference to the choice at a given day, the choices in two consecutive days, the choices over $n > 2$ consecutive days.

Computation of probabilities, transition probabilities and sequence probabilities is equivalent to evaluation of the probability measure of sets, representing events, in the space of the random terms. This follows from the set-theoretic approach to probability introduced by Kolmogorov (1956). Generally, as mentioned in the introduction, these probabilities do not satisfy the Markov property.

The probability of selecting alternative i_n at day t_n is:

$$P^r_{i_n} = \mathbb{P}\left(\Im^r(t_n) = i_n\right) = \mathbb{P}\left(S^r_{i_n}\right) \qquad r = 1, ..., R \tag{14}$$

the transition probability, i.e. the probability of selecting alternative i_{n-1} at day t_{n-1} and alternative i_n at day t_n is:

$$P^r_{i_{n-1}i_n} = \mathbb{P}\left(\Im^r(t_{n-1}) = i_{n-1}, \Im^r(t_n) = i_n\right) = \mathbb{P}\left(S^r_{i_{n-1}} \times S^r_{i_n}\right)$$

$$r = 1, ..., R \tag{15}$$

the probability of the sequence of choices $i_1, ..., i_n$ over the n consecutive days $t_1, .., t_n$ is:

$$P^r_{i_1 ... i_n} = \mathbb{P}\left(\mathfrak{I}^r(t_1) = i_1, ..., \mathfrak{I}^r(t_n) = i_n\right) = \mathbb{P}\left(S^r_{i_1} \times ... \times S^r_{i_n}\right)$$

$$r = 1, ..., R \tag{16}$$

where $\times$ denotes Cartesian product and the sets are defined by:

$$S^r_{i_k} = \left\{u^r_{i_k}(t_k) \geq u^r_j(t_k),\ j = 1, ..., J^r,\ j \neq i_k\right\} \subset \mathbb{R}^{J^r}$$

$$k = 1, ..., n \quad r = 1, ..., R \tag{17}$$

As the next section will show, transition probabilities are of particular interest in assignment processes. Computation depends on the stochastic process assumed for the random terms. In the case of assumption A2a, the one of the bi-extremal distribution, Delle Site and Salucci (2015) have shown that it is possible to express transition choice probabilities in analytic form. In the case of assumption A2b, the one of multivariate normal, computation requires simulation, i.e. drawing from the distribution of the random terms (see Delle Site and Salucci, 2013, for a review on computation of transition probabilities).

3 Assignment process and fixed points

In deterministic assignment processes, path flows F^r_i are given by the product of the OD demand q^r by the path probability P^r_i. This can be justified in two ways. With individual-specific random terms we have a continuum of individuals. Thus, probabilities are intepreted as proportions. Alternatively, the number of individuals using a path is regarded as a random variable given by the sum of q^r Bernoulli variables with success probability equal to P^r_i. The expected value of this random variable is the OD demand q^r times the probability P^r_i.

Using the same argument, we can define the transition flows $F^r_{i_{n-1} i_n}$ relating to day t_{n-1} and t_n as the product of q^r by the transition probability $P^r_{i_{n-1} i_n}$. Transition flows represent the flows of shifters and non shifters.

The following lemma provides the formulas for iterative computation of path flows on the basis of the underlying choice process.

Lemma. *Under assumptions A1, A2a or A2b, and A3, path flows at day t_n are given by:*

$$F^r_{i_n} = q^r \cdot P^r_{i_n} =$$

$$= \sum_{i_{n-1}=1}^{J^r} F^r_{i_{n-1}i_n} = q^r \cdot \sum_{i_{n-1}=1}^{J^r} P^r_{i_{n-1}i_n}$$

$$i_n = 1, ..., J^r \quad r = 1, ..., R \tag{18}$$

where:

$$P^r_{i_n} = \mathbb{P}(S^r_{i_n})$$

$$S^r_{i_n} = \left\{ v^r_{i_n}(t_n) + \varepsilon^r_{i_n}(t_n) \geq v^r_j(t_n) + \varepsilon^r_j(t_n)\,,\; j = 1, ..., J^r,\; j \neq i_n \right\} \subset \mathbb{R}^{J^r} \tag{19}$$

and

$$P^r_{i_{n-1}i_n} = \mathbb{P}(S^r_{i_{n-1}} \times S^r_{i_n})$$

$$S^r_{i_{n-1}} = \left\{ v^r_{i_{n-1}}(t_{n-1}) + \varepsilon^r_{i_{n-1}}(t_{n-1}) \geq v^r_j(t_{n-1}) + \varepsilon^r_j(t_{n-1})\,,\; j = 1, ..., J^r,\; j \neq i_{n-1} \right\} \subset \mathbb{R}^{J^r} \tag{20}$$

Proof. See Appendix 2. □

The lemma provides path flows in terms of transition flows. A direct consequence of the lemma is the following invariance property: under assumptions A1, A2a or A2b, and A3, day-to-day correlation is irrelevant to path flows at any day, while it is relevant to transition flows.

Now consider the stationary choice process. Let $P^r_i\left(\bar{F}^r_i; i = 1, ..J^r; r = 1, ..R\right)$ denote the probability as function of path flows. The following proposition provides the formulas of the associated flows and transition flows.

Proposition 1. *Under assumptions A1, A2a or A2b, and A3, the path flows associated with the stationary choice process are given by:*

$$\bar{F}^r_i = q^r \cdot P^r_i\left(\bar{F}^r_i; i = 1, ..J^r; r = 1, ..R\right) =$$

$$= \sum_{k=1}^{J^r} \bar{F}^r_{ki} = q^r \cdot \sum_{k=1}^{J^r} P^r_{ki}\left(\bar{F}^r_i; i = 1, ..J^r; r = 1, ..R\right)$$

$$i = 1, ..., J^r \quad r = 1, ..., R \tag{21}$$

Proof. See Appendix 2. □

A direct consequence of proposition 1 is the following invariance property: under assumptions A1, A2a or A2b, and A3, day-to-day correlation is irrelevant to path flows associated with the stationary choice process, while it is relevant to transition flows.

Notice that the lemma and proposition 1 can be extended to sequences of more than two days.

Flows and transition flows of the stationary choice process are the solutions of the fixed-point problem represented by eqns (21). Flows $\bar{F}_i^r$, $i = 1, ..., J^r$, $r = 1, ..., R$, are those of classical SUE (Daganzo and Sheffi, 1977). Therefore, we have the following two corollaries.

Corollary 1. *The solutions to the logit SUE are the fixed point states of a day-to-day deterministic assignment process under assumptions A1, A2a and A3.*

Corollary 1 specifies that the random term updating process from which logit SUE is derived is the stationary extremal process of eqns (9).

Corollary 2. *The solutions of the probit SUE are the fixed point states of a day-to-day deterministic assignment process under assumptions A1, A2b and A3.*

Corollary 2 specifies that the random term updating process from which probit SUE is derived is the stationary Gaussian vector process of eqns (12).

Proposition 1 and its corollaries have a significant implication in terms of how logit and probit SUE are intepreted. The condition of the network associated with the stationary choice process is one where, at macro level, the observed route flows do not change across days, while, at micro level, individuals change route. The micro-shifts are quantified by the transition flows over two consecutive days $\bar{F}_{ki}^r$. The following propositions characterise the transition flows at SUE.

Consider the matrix of transition flows $\bar{F}_{ki}^r$ for an OD pair r. We have the paths chosen the day before on the rows, and the paths chosen the day after on the columns.

Proposition 2. *Under assumptions A1, A2a or A2b, and A3, at SUE the matrix of transition flows is symmetric:*

$$\bar{F}_{ki}^r = \bar{F}_{ik}^r \quad k, i = 1, ..., J^r,\ k \neq i,\ r = 1, ..., R \tag{22}$$

Proof. See Appendix 2. □

Proposition 2 implies that the number of shifters from path k to path i equals the number of shifters from path i to path k.

Of particular interest are the cases of the following assumptions.

Assumption A2c - Random term updating model, unchanged case. *The stochastic process of random terms over consecutive days satisfies assumption A2a or A2b and, in addition, the vectors of the random terms are unchanged across days, i.e.:*

$$\varepsilon_i^r(t_1) = ... = \varepsilon_i^r(t_n) \qquad i = 1, ..., J^r,\ r = 1, ..., R \tag{23}$$

Proposition 3. *Under assumptions A1, A2c and A3, at SUE the transition flows satisfy:*

$$\bar{F}^r_{ki} = 0 \qquad k, i = 1, ..., J^r,\ k \neq i,\ r = 1, ..., R \tag{24}$$

Proof. See Appendix 2. □

Proposition 3 states that if random terms are unchanged across days, at SUE no user changes route.

Assumption A2d - Random term updating model, independence case. *The stochastic process of random terms over consecutive days satisfies assumption A2a or A2b and, in addition, the vectors of the random terms are independent across days.*

Proposition 4. *Under assumptions A1, A2d and A3, at SUE the transition flows satisfy:*

$$\bar{F}^r_{ki} = q^r \cdot P^r_k \left(\bar{F}^r_i; i = 1, ..J^r; r = 1, ..R\right) \cdot P^r_i \left(\bar{F}^r_i; i = 1, ..J^r; r = 1, ..R\right)$$

$$k, i = 1, ..., J^r,\ r = 1, ..., R \tag{25}$$

Proof. See Appendix 2. □

Proposition 4 provides the expressions of the transition flows at SUE under the independence assumption of the classical framework of traffic assignment processes.

4 Illustrative examples

4.1 Two-link network

The first example relates to a two-link network representing a town centre route and a bypass route (Figure 8). We assume a total demand q=1200 vehicles/hour. Bureau of Public Roads (BPR) volume-delay functions derived empirically for similar routes are used. The functions, in hours, are:

$$T_1 = 0.057 \cdot \left[1 + (F_1/800)^{5.2}\right] \tag{26}$$

for the town-centre route, and:

$$T_2 = 0.045 \cdot \left[1 + 0.68\,(F_2/1230)^{4.6}\right] \tag{27}$$

for the bypass route.

For route choice, we consider a random utility model with systematic utility depending on travel time only. The travel time coefficient is $\beta_T = -0.10796$, with travel time in minutes, based on the estimation of a binomial logit route choice model which has used stated preference data (Delle Site and Filippi, 2011).

Table 1 shows the values at SUE of probabilities $\bar{P}_1$ and $\bar{P}_2$, path flows $\bar{F}_1$ and $\bar{F}_2$, and travel times in the two cases of a logit and a probit route choice model. As in the logit, the probit model that has been used considers independence of the random terms across path alternatives. The variance of the random terms in the probit is equal to unity. The values in the table are obtained by solving the fixed point problem of eqns (21) with respect to the path flows. In the probit, we have kept unchanged the value of the time coefficient to enable comparison between the two distributions of the random terms.

For logit, probabilities are computed using the closed-form expressions:

$$\bar{P}_i = \frac{e^{v_i}}{e^{v_1} + e^{v_2}} \quad i = 1, 2 \tag{28}$$

For probit, probabilities are also computed analytically using the expressions that hold in the binomial case (Ben-Akiva and Lerman, 1985):

$$\bar{P}_i = \Phi\left(\frac{v_i - v_j}{\sqrt{2}}\right) \quad i, j = 1, 2 \;\; i \neq j \tag{29}$$

where Φ is the cumulative standard normal distribution.

The values of transition flows at SUE depend, according to eqns (21), on the marginal distribution of the random terms over two consecutive days.

In the logit case, where the random term of each alternative follows a bi-extremal distribution, it is possible to use the analytic expressions of transition probabilities that are found in Delle Site and Salucci (2015; eqns 44 and 45). In the present case, where systematic utilities are unchanged across choices, these reduce to the following simple expressions:

$$\bar{P}_{12} = (1 - \phi) \cdot \frac{e^{v_1}}{e^{v_1} + e^{v_2}} \cdot \frac{e^{v_2}}{e^{v_1} + e^{v_2}} \tag{30}$$

$$\bar{P}_{11} = \bar{P}_1 - \bar{P}_{12} \tag{31}$$

Figure 9 shows the linear variation of the transition flows with the parameter ϕ controlling day-to-day correlation of the bi-extremal distribution. The figure shows the values of $\bar{F}_{11}$, representing the number of individuals who stay on the town centre route, $\bar{F}_{22}$, representing the number of individuals who stay on the bypass route, and $\bar{F}_{12}$, representing the number of individuals who shift from the town-centre route to the bypass route which equals, due to the symmetry property of proposition 2, the number $\bar{F}_{21}$ of individuals who shift from the bypass route to the town-centre route.

The number of shifters $\bar{F}_{12}$ equals the product $q \cdot \bar{P}_1 \cdot \bar{P}_2 = 1200 \cdot 0.468 \cdot 0.532 = 299$ vehicles/hour in the independence case (correlation parameter equal to zero), and decreases to zero in the perfect correlation case (correlation parameter equal to unity and unchanged random terms), according to propositions 3 and 4. For any value of the parameter ϕ controlling the day-to-day correlation, the sum $\bar{F}_{11} + \bar{F}_{12}$ equals the flow $\bar{F}_1 = 562$ vehicles/hour that is found at any day on route 1, while $\bar{F}_{22} + \bar{F}_{21}$ equals the flow $\bar{F}_2 = 638$ vehicles/hour that is found at any day on route 2.

In the probit case, transition probabilities need to be computed using simulation. The frequency estimator is used:

$$\bar{P}_{ki} \cong \frac{1}{D} \cdot \sum_{d=1}^{D} I\left([\epsilon_{1d}(t_{n-1}), \epsilon_{2d}(t_{n-1}), \epsilon_{1d}(t_n), \epsilon_{2d}(t_n)]' \in S_k \times S_i\right) \quad k, i = 1, 2 \tag{32}$$

where $I(\cdot)$ is the indicator function, d the draw index and D the number of draws (10^6 in the case here). This estimator is referred to by Train (2003) as accept-reject simulator because it equals the proportion of draws that are "accept" with respect to the transition regions of the 4-dimensional Euclidean space $\mathbb{R}^4$ of the day-before and day-after random terms. The frequency estimator is minimum variance unbiased and strongly consistent (Lerman and Manski, 1981).

Figure 10 shows the variation of the transition flows with the correlation coefficient ρ in the probit case. Notice the non-linear relationship. The same comments as in the logit apply as to the symmetry of the transition flows and the values taken in the independence and the unchanged cases.

4.2 Five-link network

The second example relates to the five-link network whose topology is shown in the graph of Figure 11. The network is found in Cascetta (2009). Its topology is the one of the Braess network (Braess, 1969; Braess et al., 2005). The network includes four nodes, five directed links and three OD pairs. The link-path incidence relationship is shown in Table 2. There is a total of six paths. OD pair (1,4) has three paths, OD pair (2,4) two paths, OD pair (3,4) one path. The OD demand flows (in vehicles/hour) are $q_{14} = 1000$, $q_{24} = 1500$, $q_{34} = 800$. The following BPR volume-delay functions (time in minutes, flow in vehicles/hour) are assumed:

$$T_a = T_0 \cdot [1 + a \cdot (z_a/c)^\gamma] \tag{33}$$

The values of the coefficients are in Table 3.

The MSA algorithm is used to find travel times and flows at SUE. Two route choice models are considered: one logit and one probit. In both the logit and the probit model the value of the time coefficient β_T is -0.03334 (time in minutes). While in the logit model the random terms are i.i.d. across alternatives, in the probit model we assumed the inter-alternative variance-covariance matrix shown in Table 4. This is because probit offers the opportunity to model correlation of overlapping paths (path 1 is overlapping with path 2 due to the common link 1, path 2 is overlapping with path 3 due to the common link 5).

The link travel times and flows at SUE are in Table 5. The path travel times, probabilities and flows at SUE are in Table 6. For probit, probabilities are computed by simulation using the frequency estimator. Draws for the random terms of OD pair (1,4), which includes the three inter-correlated paths, are obtained based on the Cholesky decomposition method recalled in section 2.2.2 (Scheuer and Stoller, 1962).

The transitions at SUE for the OD pair (1,4) are illustrated in the transition graph of Figure 12. The nodes in the transition graph denote the paths of the OD pair. The links denote the transitions from one path to another. The loops denote the permanence on the path. The transition flow matrix is shown in Table 7 for logit ($\phi = 0.5$), and in Table 8 for probit (within-alternative day-to-day correlation ρ equal to 0.5). Transition probabilities are computed for both logit and probit by simulation using the frequency estimator on the basis of, respectively, eqns (9) and eqns (12). The symmetry property of the transition flows at SUE, which is a theoretical property based on proposition 2, is confirmed numerically.

5 Conclusion

The framework of deterministic assignment processes that the paper has outlined builds on three pillars: the deterministic sequence of systematic utilities, the stochastic process of random terms and the resulting stochastic process of choice. Day-to-day correlation of the random terms provides the foundation for realistic representation of persistence of habits and unobservables that characterises route choice behaviour. SUE are shown to be the fixed point states of the stationary choice process.

The paper has provided a novel interpretation of classical SUE based on the explicit treatment of transition flows. At SUE, the observed path flows do not change, but day-to-day shifts between routes occur. Only if random terms are unchanged across days, SUE can be seen as a condition of "rest" at both the macro and the micro level. Moreover, transition flows at SUE satisfy a symmetry condition.

The circumstance that path flows that are obtained at any day and those at SUE are invariant with the day-to-day correlation of the random terms has a remarkable implication related to the validity of the results in the literature on the stability of deterministic assignment processes. The implicit Markovian assumption for the choice process and the independence assumption characterising the classical framework are irrelevant to the stability results which appear to have more general validity.

The framework provided is admittedly in an initial stage when compared with the behavioural assumptions developed for deterministic assignment processes. Thus, several issues deserve further research.

One relates to the simple learning filter of the systematic utility updating model. This has been restricted to one past day only for consistency with the random term updating model which we wanted to maintain simple for convenience. From the literature reviewed, we could not find ready solutions to the problem consisting in the identification of autoregressive structures that have higher order in terms of past days and, at the same time, are able to save stationarity (i.e. constancy of the one-day, two-day ... marginal probability distributions). Solving this problem requires a statistical investigation which is left for future research.

The learning filter should also be addressed to accommodate the presence of ATIS providing the users with information on current travel times (see Bifulco et al., 2016, for a recent contribution). An attendant development is the extension to multiple user classes, to account for heterogeneity in the level of knowledge of network travel time conditions and/or in the marginal disutility of travel time. Still for the systematic utility updating model, it might be possible to consider inertia effects in the form of state dependence (as in Cascetta and Cantarella, 1991), a behaviour where individuals pay a psychological extra-cost if they change route. This extension appears not trivial being a case where the invariance property related to the day-to-day correlation proved in the paper does not hold.

Additionally, the random term updating model. The undoubtedly unrealistic behaviour postulated in the extremal process, based on the maximum operator, is essentially motivated by the desire to justify the use of logit SUE in network analysis, with the related advantages. The behaviour postulated in the Gaussian vector process appears more realistic since it is based on the additive structure of the autoregression. Future research might investigate the stationary processes associated with the random terms of other path choice models, such as the (multiplicative) closed-form weibit model (Castillo et., 2008), based on independent heteroschedastic Weibull distributions with variance dependent on path cost, and the (additive) gammit model (Cantarella and Binetti, 2002), based on the multivariate gamma function, which provides the same flexibility in the correlation pattern as the probit.

The framework developed can be transferred, with proper modifications, from route choice to the choice of the transport mode (a mode-choice version of SUE is dealt with in Cantarella et al., 2015).

Appendix 1. Classical framework of deterministic assignment processes.

This appendix aims to make evident that the classical framework of deterministic assignment processes builds on the formulation of a choice updating process with an implicit Markovian assumption. Also, we show that the framework, in its most frequent formulation, implies day-to-day independence of the random terms.

First, the framework assumes for the choice updating process the following equality (reference is made to section 1.3 “Users’ choice behavior modeling” in Cantarella and Cascetta, 1995):

$$P_{i_n} = \sum_{i_{n-1}} P_{i_{n-1}} \cdot P_{i_n/i_{n-1}} \tag{34}$$

where P_{i_n} is the probability of choosing path i_n at day t_n, $P_{i_{n-1}}$the probability of choosing path i_{n-1} at day t_{n-1}, and $P_{i_n/i_{n-1}}$ the probability of choosing path i_n conditional on having chosen path i_{n-1} the previous day.

The equality is simply an application of the law of total probability. With eqn (34) only, the probability distribution of the path choice over a sequence of days remains unspecified. As elucidated by Watling and Cantarella (2013; section 2.2 "Representation of state and distribution"), to derive the probability of the sequence of choices, appeal is made to a Markovian assumption whereby (first-order Markov chain):

$$P_{i_n/i_{n-1}} = P_{i_n/i_1 \ldots i_{n-1}} \tag{35}$$

where $P_{i_n/i_1 \ldots i_{n-1}}$ is the probability of choosing path i_n conditional on having chosen path i_1 the first day,..., path i_{n-1} at day t_{n-1}.

Eqn (35) implies the following factoring property for the sequence probability (theorem 1.1.1 in Norris, 1997):

$$P_{i_1 i_2 \ldots i_{n-1} i_n} = P_{i_1} \cdot P_{i_2/i_1} \cdot \ldots \cdot P_{i_{n-1}/i_{n-2}} \cdot P_{i_n/i_{n-1}} \tag{36}$$

where $P_{i_1 i_2 \ldots i_{n-1} i_n}$ is the probability of choosing path i_1 the first day, path i_2 at day t_2, ... , path i_{n-1} at day t_{n-1}, and path i_n at day t_n; P_{i_1} the probability of choosing i_1 the first day; and with obvious meaning of the other symbols.

Second, the framework is usually formulated according to the assumption that conditional probabilities are independent of the path chosen in the previous day:

$$P_{i_n/i_{n-1}} = P_{i_n} \tag{37}$$

The assumption in eqn (37) holds if random terms are day-to-day independent and if there is no path dependence of the systematic utilities[5].

The assumption in eqn (37) yields for the probability of the sequence (i_{n-1}, i_n):

$$P_{i_{n-1} i_n} = P_{i_{n-1}} \cdot P_{i_n/i_{n-1}} = P_{i_{n-1}} \cdot P_{i_n} \tag{38}$$

Eqn (38) expresses the transition probability as the product of the probability the day before times the probability the day after.

[5]This excludes state-dependent route choice models whereby the currently used path is assigned extra-utility.

Appendix 2. Proofs.

Lemma

Consider day 1. Systematic utilities are, by assumption, given (they can be those of travel times at zero flows). Path choice probabilities at day 1 can be computed on the basis of the given systematic utilities and the multivariate marginal distribution of the random terms. Path flows, and associated travel times, follow.

Now consider day 2. Choices are made on the basis of systematic utilities which depend on the travel times of day 1. Path choice probabilities at day 2 can be computed on the basis of the given systematic utilities and the multivariate marginal distribution of the random terms. In fact, only the random terms of day 2 enter the computation of probabilities at day 2, random terms at day 1 are irrelevant, as it is evident by the definition of the set $S^r_{i_n}$ of eqn (19). The argument applies to any following day. This proves the first equality of eqns (18).

The expression of path flows in terms of transition flows is consequence of the law of total probability applied to the choice process. In the light of the first part of proposition 1, the systematic utilities are known up to day t_n and the transition flows depend on the marginal distribution of the random terms over the two days t_{n-1} and t_n, as it is evident by the definitions of the sets $S^r_{i_{n-1}}$ and $S^r_{i_n}$. □

Proposition 1.

This follows from the lemma and application of the definition of strict stationarity to the choice process. □

Proposition 2.

We use the notation:

$$F^r_{ik} = q^r \cdot P^r_{ki}(\bar{F}^r_i;\ i = 1, ..J^r;\ r = 1, ..R) = q^r \cdot \bar{P}^r_{ki} \tag{39}$$

Consider the transition probability:

$$\bar{P}^r_{ki} = \mathbb{P}\left(\mathfrak{I}^r\left(t_{n-1}\right) = k_{n-1}, \mathfrak{I}^r\left(t_n\right) = i_n\right) = \mathbb{P}(S^r_{k_{n-1}} \times S^r_{i_n}) \tag{40}$$

This is given by the following multiple integral in the $\mathbb{R}^{2 \cdot J^r}$ space:

$$\bar{P}^r_{ki} = \int ... \int_{S^r_{k_{n-1}} \times S^r_{i_n}} f\left[\boldsymbol{\varepsilon}\left(t_{n-1}\right), \boldsymbol{\varepsilon}\left(t_n\right)\right] d\boldsymbol{\varepsilon}\left(t_n\right) d\boldsymbol{\varepsilon}\left(t_{n-1}\right) \tag{41}$$

which can be computed as iterated integral as follows:

$$\bar{P}_{ki}^{r} = \int \cdots \int_{S_{k_{n-1}}^{r}} \left[\int \cdots \int_{S_{i_n}^{r}} f\left[\boldsymbol{\varepsilon}\left(t_{n-1}\right), \boldsymbol{\varepsilon}\left(t_n\right)\right] d\boldsymbol{\varepsilon}\left(t_n\right) \right] d\boldsymbol{\varepsilon}\left(t_{n-1}\right) \tag{42}$$

where $\boldsymbol{\varepsilon}\left(t_{n-1}\right) = \left[\varepsilon_j\left(t_{n-1}\right), j = 1, ..J^r\right]'$, $\boldsymbol{\varepsilon}\left(t_n\right) = \left[\varepsilon_j\left(t_n\right), j = 1, ..J^r\right]'$, $f\left[\cdot\right]$ is the p.d.f.. Consider now the transition probability:

$$\bar{P}_{ik}^{r} = \mathbb{P}\left(\mathfrak{I}^r\left(t_{n-1}\right) = i_{n-1}, \mathfrak{I}^r\left(t_n\right) = k_n\right) = \mathbb{P}(S_{i_{n-1}}^{r} \times S_{k_n}^{r})$$

which is given by the following iterated integral:

$$\bar{P}_{ik}^{r} = \int \cdots \int_{S_{i_{n-1}}^{r}} \left[\int \cdots \int_{S_{k_n}^{r}} f\left[\boldsymbol{\varepsilon}\left(t_{n-1}\right), \boldsymbol{\varepsilon}\left(t_n\right)\right] d\boldsymbol{\varepsilon}\left(t_n\right) \right] d\boldsymbol{\varepsilon}\left(t_{n-1}\right) \tag{43}$$

By Fubini's theorem (theorem 5.2.2 in Athreya and Lahiri, 2010) we can exchange the order of integration and get:

$$\bar{P}_{ik}^{r} = \int \cdots \int_{S_{k_n}^{r}} \left[\int \cdots \int_{S_{i_{n-1}}^{r}} f\left[\boldsymbol{\varepsilon}\left(t_{n-1}\right), \boldsymbol{\varepsilon}\left(t_n\right)\right] d\boldsymbol{\varepsilon}\left(t_{n-1}\right) \right] d\boldsymbol{\varepsilon}\left(t_n\right) \tag{44}$$

We shall prove that $\bar{P}_{ki}^{r} = \bar{P}_{ik}^{r}$ by proving that the integral of eqn (42) and the integral of eqn (44) have the same value.

Recall the following assumptions of the stationary state. First, the marginal p.d.f. at day t_{n-1}:

$$f\left[\boldsymbol{\varepsilon}\left(t_{n-1}\right)\right] = \int \cdots \int_{\mathbb{R}^{J^r}} f\left[\boldsymbol{\varepsilon}\left(t_{n-1}\right), \boldsymbol{\varepsilon}\left(t_n\right)\right] d\boldsymbol{\varepsilon}\left(t_n\right) \tag{45}$$

and the marginal p.d.f. at day t_n:

$$f\left[\boldsymbol{\varepsilon}\left(t_n\right)\right] = \int \cdots \int_{\mathbb{R}^{J^r}} f\left[\boldsymbol{\varepsilon}\left(t_{n-1}\right), \boldsymbol{\varepsilon}\left(t_n\right)\right] d\boldsymbol{\varepsilon}\left(t_{n-1}\right) \tag{46}$$

have the same functional form.

Second, the systematic utilities of each alternative are constant across days, which implies that $S_{k_{n-1}}$and $S_{k_n}$are identical sets, and also $S_{i_{n-1}}$and $S_{i_n}$are identical sets. The consequence is that the marginal p.d.f. at day t_{n-1} when i is chosen at day t_n:

$$f_{,i}\left[\boldsymbol{\varepsilon}\left(t_{n-1}\right)\right] = \int \cdots \int_{S_{i_n}^{r}} f\left[\boldsymbol{\varepsilon}\left(t_{n-1}\right), \boldsymbol{\varepsilon}\left(t_n\right)\right] d\boldsymbol{\varepsilon}\left(t_n\right) \tag{47}$$

and the marginal p.d.f. at day t_n when i is chosen at day t_{n-1}:

$$f_{i,}\left[\boldsymbol{\varepsilon}\left(t_n\right)\right] = \int \cdots \int_{S_{i_{n-1}}^{r}} f\left[\boldsymbol{\varepsilon}\left(t_{n-1}\right), \boldsymbol{\varepsilon}\left(t_n\right)\right] d\boldsymbol{\varepsilon}\left(t_{n-1}\right) \tag{48}$$

have the same functional form.

Thus, the integral of eqn (42) which provides $\bar{P}^r_{ki}$ can be re-written as:

$$\bar{P}^r_{ki} = \int ... \int_{S^r_{k_{n-1}}} f_{,i} \left[\boldsymbol{\varepsilon} \left(t_{n-1} \right) \right] d\boldsymbol{\varepsilon} \left(t_{n-1} \right) \tag{49}$$

Also, the integral of eqn (44) which provides $\bar{P}^r_{ik}$ can be re-written as:

$$\bar{P}^r_{ik} = \int ... \int_{S^r_{k_n}} f_{i,} \left[\boldsymbol{\varepsilon} \left(t_n \right) \right] d\boldsymbol{\varepsilon} \left(t_n \right) \tag{50}$$

Since the only difference is in the integration variables, the two integrals of eqn (49) and (50) have the same value. □

Proposition 3.

Under assumption A2c, the space of the random terms where the transition probabilities are computed reduces from dimension $2 \cdot J^r$ to J^r. We have $S^r_k \cap S^r_i = \emptyset$ since, for each vector of random terms, one alternative only is selected. Therefore, $\bar{P}^r_{ki} = 0$, $k \neq i$. □

Proposition 4.

Under assumption A2d the random terms are independent, therefore, the transition probability from k to i equals the probability of choosing k the day before times the probability of choosing i the day after. □

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Table 1: Probabilities, flows and travel times at SUE

	logit		probit	
	town centre route	bypass route	town centre route	bypass route
probability	0.468	0.532	0.465	0.535
flow (vehicles/hour)	562	638	558	642
travel time (minutes)	3.96	2.79	3.95	2.79

Table 2: Link-path incidence relationship

OD pair	path	link sequence
(1,4)	1	1-4
	2	1-3-5
	3	2-5
(2,4)	4	3-5
	5	4
(3,4)	6	5

Table 3: Coefficients of the volume-delay functions

link	T_0	a	c	γ
1	10	2	1000	4
2	22	2	1000	4
3	13	2	2500	4
4	20	2	1000	4
5	11	2	3300	4

Table 4: Probit case: one-day variance-covariance matrix for OD pair (1,4)

path	1	2	3
1	1	0.2	0
2	0.2	1	0.2
3	0	0.2	1

Table 5: Link travel times and flows at SUE

link	logit travel time (minutes)	logit flow (vehicles/hour)	probit travel time (minutes)	probit flow (vehicles/hour)
1	12.6	599	12.2	579
2	23.1	401	23.4	421
3	14.5	1232	14.5	1221
4	42.6	867	41.6	858
5	17.5	2433	17.6	2442

Table 6: Path travel times, probabilities and flows at SUE

path	logit travel time (minutes)	logit probability	logit flow (vehicles/hour)	probit travel time (minutes)	probit probability	probit flow (vehicles/hour)
1	55.1	0.247	247	53.9	0.241	241
2	44.6	0.351	352	44.3	0.338	338
3	40.6	0.401	401	41.0	0.421	421
4	32.0	0.587	881	32.1	0.590	884
5	42.6	0.413	619	41.6	0.410	616
6	17.5	1	800	17.6	1	800

Table 7: Transition flows (in vehicles/hour) for OD pair (1,4) in logit SUE

path	1	2	3
1	154	43	50
2	43	238	70
3	50	70	281

Table 8: Transition flows (in vehicles/hour) for OD pair (1,4) in probit SUE

path	1	2	3
1	111	60	70
2	60	181	97
3	70	97	254

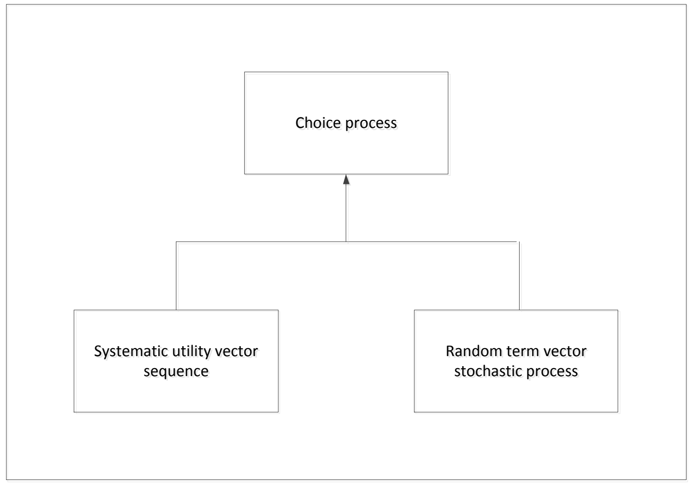

Figure 1: The framework pillars

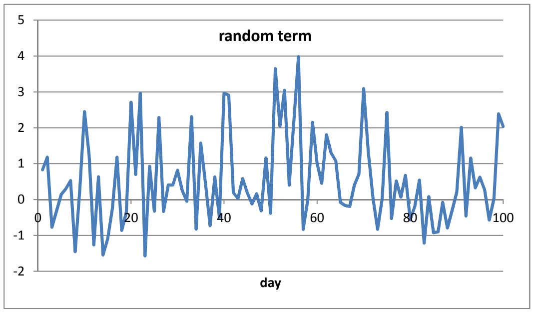

Figure 2: Univariate extremal process: $\phi = 0$

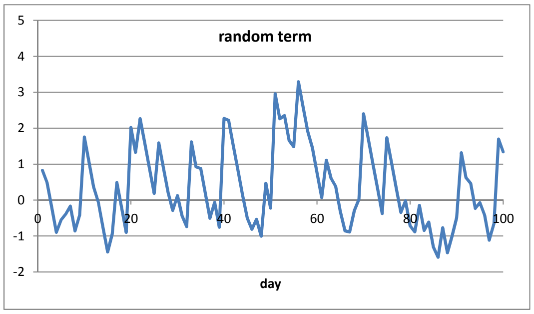

Figure 3: Univariate extremal process: $\phi = 0.5$

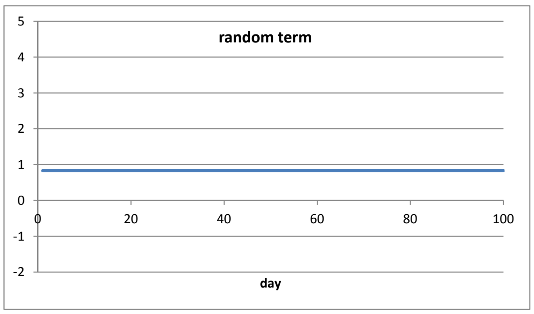

Figure 4: Univariate extremal process: $\phi = 1$

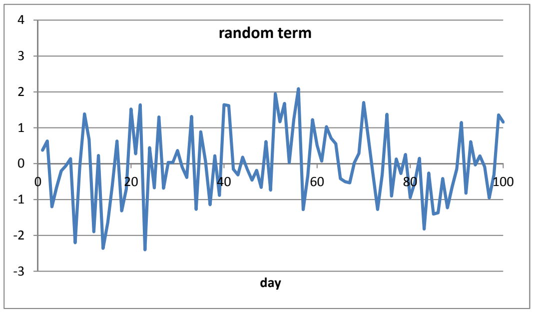

Figure 5: Univariate Gaussian process: $\rho = 0$

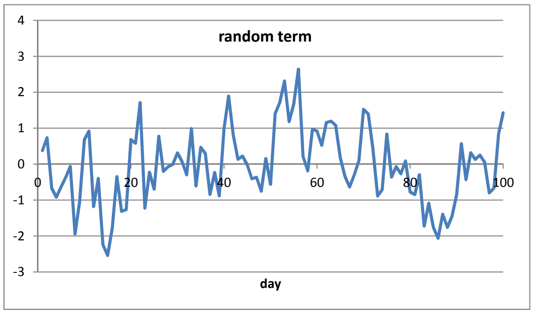

Figure 6: Univariate Gaussian process: $\rho = 0.5$

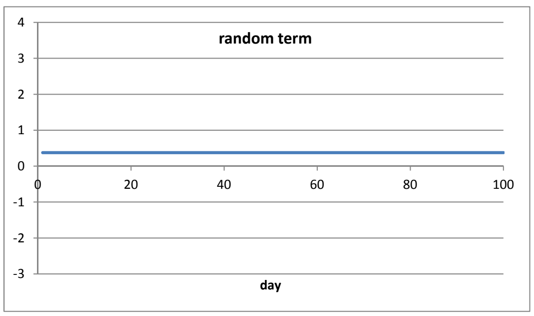

Figure 7: Univariate Gaussian process: $\rho = 1$

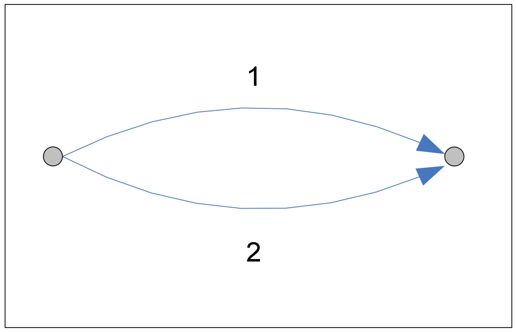

Figure 8: Two-link network

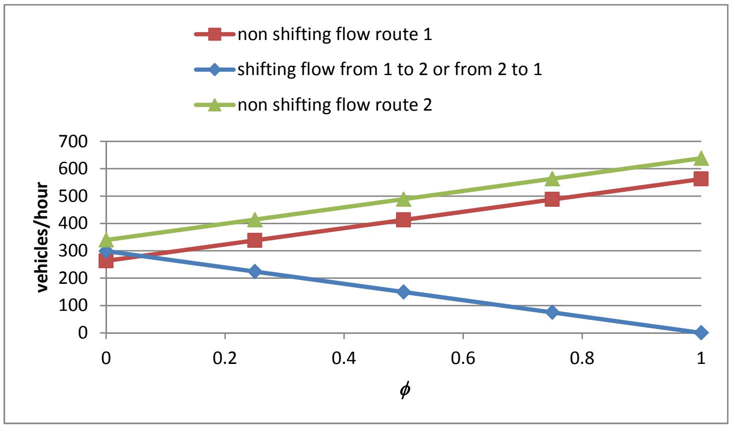

Figure 9: Transition flows in logit SUE

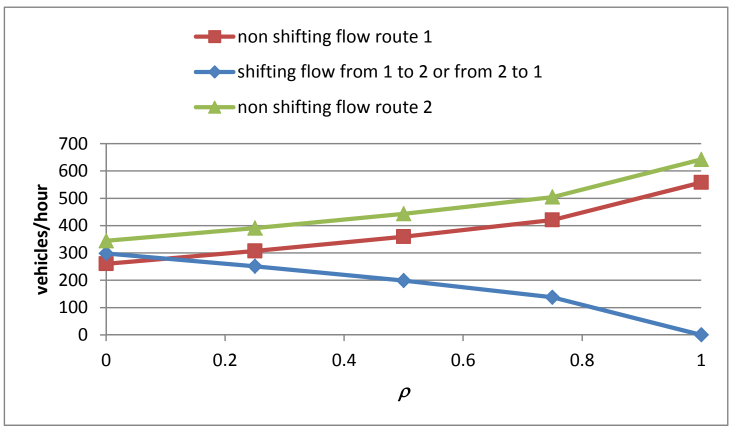

Figure 10: Transition flows in probit SUE

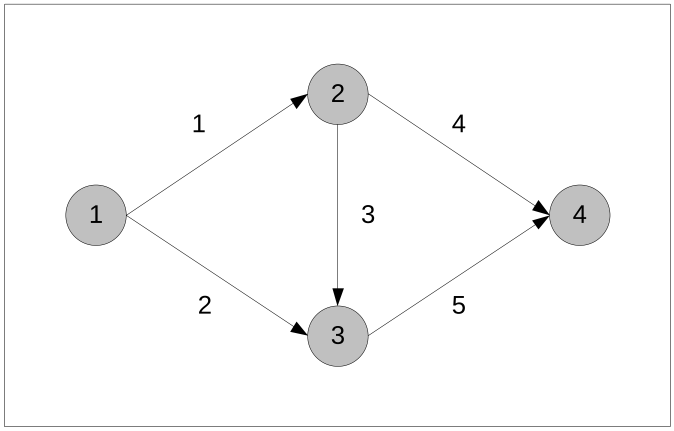

Figure 11: Five-link network

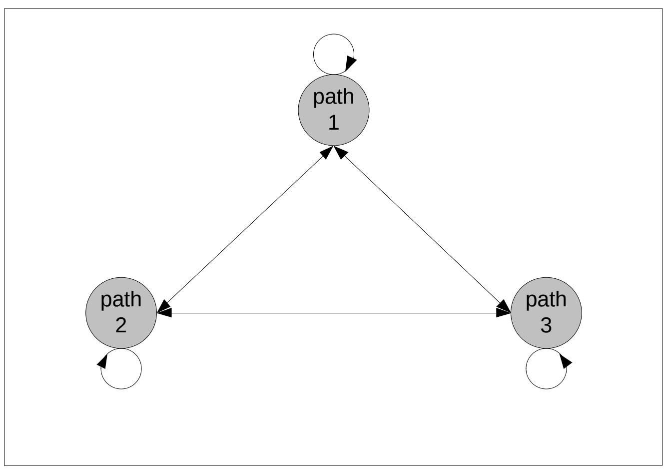

Figure 12: Transition graph for OD pair (1,4)